Influence Coefficients of a Circular Cylindrical Shell with Rapidly Varying Parabolic Wall Thickness

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A closed-form solution is given of the differential equation defining the axisymmetric deformations of a circular cylindrical shell with parabolically varying wall thickness, and the 16 influence coefficients connecting the deformations and the stress resultants acting in the end sections of a shell of finite length are presented in tabular form. A numerical example shows that the maximal stresses calculated with the aid of these influence coefficients can in some cases differ significantly from those obtained when the stress analysis is carried out in the usual manner, that is, with the aid of the influence coefficients of the shell of uniform wall thickness.

1. Introduction

A COMMON structure in the missile industry is the circular cylindrical shell whose wall thickness is constant. This shell supports the various loads imposed upon it principally by means of membrane stresses that are constant through the wall thickness. However, in many structural applications, it is necessary to join the cylinder to a shell of unequal radius of curvature or to a rigid bulkhead. Within a distance or boundary layer whose length from the juncture is of order

$$l = (ah_0)^{1/2} \tag{1.1}$$

where a is the radius and h_0 is the thickness of the shell, significant bending stresses and local variations of the membrane stresses arise; these are usually called discontinuity stresses. To strengthen the structure in this region, the wall thickness is often increased locally. The effect of this strengthening is generally not taken into account in the analysis of the stresses, because influence coefficients for finite-length circular cylindrical shells with rapidly varying wall thickness have not been calculated and tabulated. The purpose of the present paper is to help eliminate this inconsistency from the analysis through the development of a closed-form solution of the governing equation for shells with parabolically varying wall thickness and the tabulation of influence coefficients of cylindrical shells whose wall thickness varies considerably within the characteristic length (1.1).

Various authors have considered rotationally symmetric shells with variable wall thickness. Timoshenko[1] and Flügge[2] investigated a dome loaded by its own weight. They calculated the shape of the meridian such that the membrane stresses are constant everywhere and showed that for such a dome the thickness of the wall varies exponentially. De Silva and Naghdi[3] examined a class of shells of revolution whose wall thickness varies in such a way that the two governing differential equations of the problem (moment equilibrium and compatibility) can be combined into a single second-order complex differential equation and solved by a method of asymptotic integration. Honegger[4] solved the problem of a conical shell with linearly varying wall thickness.

Many papers have been written on the circular cylindrical shell with variable wall thickness. Reissner and Sledd[5] determined the upper and lower bounds of the influence coefficients of shells with arbitrarily varying thickness by application of the minimal principles of elasticity. They investigated the special case of a semi-infinite cylinder with exponentially varying thickness. C. R. Steele,[6] in some unpublished calculations, found the exact solution of the equation of equilibrium of such a shell and showed that the exact values of the influence coefficients are about midway between the upper and lower bounds. Meissner[7] analyzed a cylindrical shell whose wall thickness varies linearly, and Timoshenko[1] and Flügge[2] computed the discontinuity stresses in such shells where the thickness varies slowly enough that the effect of moments and transverse shear forces applied at one end is not felt at the other end. Federhofer[8] considered the case of parabolically varying wall thickness and solved the problem of a liquid-filled tank whose upper boundary is free and whose lower boundary is clamped. The wall thickness varies along the entire length of the tank. He also calculated stresses and displacements in such a tank loaded axisymmetrically along its lower boundary by a moment or a shear force. However, he did not develop closed-form expressions for the influence coefficients, as is done here.

Esslinger[9] derived influence coefficients for a finite-length cylindrical shell of constant thickness. These quantities are rederived herein in a different form that allows them to be easily compared to the influence coefficients of a cylindrical shell with parabolically varying wall thickness.

Sledd[10] derived influence coefficients for circular cylindrical shells with linearly varying wall thickness.

In this paper, the influence coefficients of a finite circular cylindrical shell whose wall thickness varies rapidly and quadratically are obtained in closed form, and these influence coefficients are employed to find the discontinuity stresses in a cylindrical shell thickened only in the boundary region near the clamped edge at the bottom.

Received March 9, 1964; revision received June 19, 1964. The work here presented is part of a thesis submitted to Stanford University in partial fulfillment of the requirements for the Ph.D. degree.

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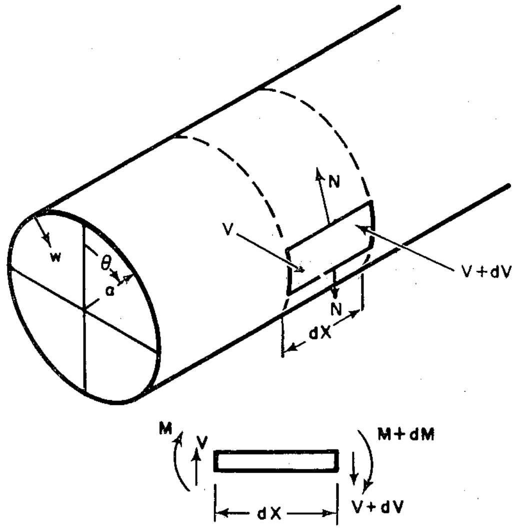

Fig. 1 Sign convention.

2. Differential Equation of Equilibrium and Solution

The equation of equilibrium of a circular cylindrical shell that deforms symmetrically to its axis is given by Timoshenko[1] as

$$\frac{d^2}{dx^2}\left(D\frac{d^2w}{dx^2}\right) + \frac{Eh}{a^2}w = 0 \qquad (2.1)$$

where D is the flexural rigidity

$$D = Eh^3/12(1 - \nu^2) \qquad (2.2)$$

The sign convention for the stress and displacement quantities is shown in Fig. 1. The following nondimensional quantities will be introduced in this equation:

$$H = h/h_0 \qquad \xi = x/x_1 \qquad \delta = w/a \qquad (2.3)$$

Here h_0 is the thickness of the shell at $x = x_1$, that is, at $\xi = 1$ (see Fig. 2).

The thickness of the shell is assumed to vary parabolically:

$$H = b\xi^2 \qquad (2.4)$$

Since $h = h_0$ at $x = x_1$, the constant b must be equal to unity. Substitution from Eqs. (2.3) and (2.4) into Eq. (2.1) gives

$$(\xi^6\delta'')'' + 4c^4\xi^2\delta = 0 \qquad (2.5)$$

where

$$(\ )' \equiv d(\ )/d\xi \qquad c \equiv \tau x_1/a$$
$$\tau \equiv [3(1 - \nu^2)(a/h_0)^2]^{1/4} \qquad (2.6)$$

The solution of Eq. (2.5) has the form

$$\delta = \xi^p \qquad (2.7)$$

which yields the characteristic equation

$$p(p - 1)(p + 3)(p + 4) + 4c^4 = 0 \qquad (2.8)$$

Substitution of the new variable

$$q = p + \tfrac{3}{2} \qquad (2.9)$$

eads to

$$(q^2 - \tfrac{9}{4})(q^2 - \tfrac{25}{4}) + 4c^4 = 0 \qquad (2.10)$$

The four solutions of Eq. (2.10) are

$$q = \pm\alpha \pm i\beta \qquad (2.11)$$

where

$$\alpha = \left[\frac{(15^2 + 8^2c^4)^{1/2} + 17}{8}\right]^{1/2} \qquad (2.12)$$

$$\beta = \left[\frac{(15^2 + 8^2c^4)^{1/2} - 17}{8}\right]^{1/2} \qquad (2.13)$$

If c is greater than unity, the quantities α and β are both real. If c is equal to unity, β is equal to zero, and the characteristic equation (2.10) has only two distinct roots. If c is less than unity, β is imaginary, and Eq. (2.10) has four distinct real roots. If $c \neq 1$ the roots of Eq. (2.8) are

$$p_j = -\tfrac{3}{2} \pm \alpha \pm i\beta \qquad (2.14)$$

and the general solution of Eq. (2.5) becomes

$$\delta = \sum_{j=1}^{4} A_j^*\xi^{p_j} \qquad (2.15)$$

Equation (2.15) has four arbitrary constants A_j^*. These are determined from boundary conditions at x_1 and x_2 (see Fig. 2). The unique solution of Eq. (2.5) corresponding to particular boundary conditions is found more easily if Eq. (2.15) is written in the form

$$\delta = \xi^{-3/2}[A\sin\theta\sinh\phi + B\sin\theta\cosh\phi + C\cos\theta\sinh\phi + D\cos\theta\cosh\phi] \qquad (2.16)$$

where

$$\theta \equiv \beta\log\xi \qquad \phi \equiv \alpha\log\xi \qquad (2.16a)$$

If $c = 1$, the general solution of Eq. (2.5) is

$$\delta = \xi^{-3/2}[A_1\xi^{\sqrt{17}/2} + A_2\xi^{-\sqrt{17}/2} + A_3(\log\xi)\xi^{\sqrt{17}/2} + A_4(\log\xi)\xi^{-\sqrt{17}/2}] \qquad (2.17)$$

The influence coefficients will be calculated assuming that $c \neq 1$. The values for $c = 1$ may be obtained from these by allowing β to approach zero in Eqs. (3.22) to (3.31).

3. Calculation of Influence Coefficients

The moment distribution in the segment x_1 to x_2 can be calculated by substituting Eq. (2.16) in the expression

$$M = -\frac{D_0 a}{x_1^2}\xi^6\delta'' \qquad (3.1)$$

and the shear distribution can be calculated from

$$V = \frac{1}{x_1}(M)' = -\frac{D_0 a}{x_1^3}(\xi^6\delta'')' \qquad (3.2)$$

where

$$D_0 = \frac{Eh_0^3}{12(1 - \nu^2)} \qquad (3.3)$$

Differentiation of Eq. (2.16) and substitution into Eq. (3.1) and Eq. (3.2) gives

$$M = (-D_0a/x_1^2)\xi^{5/2}[A'\sin\theta\sinh\phi + B'\sin\theta\cosh\phi + C'\cos\theta\sinh\phi + D'\cos\theta\cosh\phi] \qquad (3.4)$$

$$V = (-D_0a/x_1^3)\xi^{3/2}[(\alpha B' - \beta C')\sin\theta\sinh\phi + (\alpha A' - \beta D')\sin\theta\cosh\phi + (\beta A' + \alpha D')\cos\theta\sinh\phi + (\beta B' + \alpha C')\cos\theta\cosh\phi] + \tfrac{5}{2}M/\xi x_1 \qquad (3.5)$$

The integration constants A', B', C', and D' are linear combinations of A, B, C, and D. The boundary conditions at $x = x_1$, that is at $\xi = 1$, yield

$$D' = -(x_1^2/D_0a)M_1 \qquad (3.6)$$

$$\beta B' + \alpha C' = -\frac{x_1^3}{D_0 a}\left(V_1 - \frac{5}{2}\frac{M_1}{x_1}\right) \quad (3.7)$$

The boundary conditions at $x = x_2$, that is at $\xi = x_2/x_1$, give

$$A' \sin\theta_2 \sinh\phi_2 + B' \sin\theta_2 \cosh\phi_2 + C' \cos\theta_2 \sinh\phi_2 + D' \cos\theta_2 \cosh\phi_2 = -\frac{a}{D_0}\left(\frac{x_1}{a}\right)^2\left(\frac{x_1}{x_2}\right)^{5/2} M_2 \quad (3.8)$$

$$(\alpha B' - \beta C') \sin\theta_2 \sinh\phi_2 + (\alpha A' - \beta D') \sin\theta_2 \cosh\phi_2 + (\beta A' + \alpha D') \cos\theta_2 \sinh\phi_2 + (\beta B' + \alpha C') \cos\theta_2 \cosh\phi_2 = -\frac{a^2}{D_0}\left(\frac{x_1}{a}\right)^3\left(\frac{x_1}{x_2}\right)^{3/2}\left(V_2 - \frac{5}{2}\frac{M_2}{x_2}\right) \quad (3.9)$$

where

$$\theta_2 \equiv \beta \log(x_2/x_1) \qquad \phi_2 \equiv \alpha \log(x_2/x_1) \quad (3.10)$$

The four equations, i.e., Eqs. (3.6–3.9), can be solved for A', B', C', D':

$$A' = \frac{a}{D_0}\left(\frac{x_1}{a}\right)^2 \frac{1}{\Delta}\left\{M_1\left[-\alpha\beta(\sin^2\theta_2 + \sinh^2\phi_2) + \frac{5}{2}\beta \sinh\phi_2 \cosh\phi_2 - \frac{5}{2}\alpha \sin\theta_2 \cos\theta_2\right] + V_1x_1[-\beta \sinh\phi_2 \cosh\phi_2 + \alpha \sin\theta_2 \cos\theta_2] + M_2\left(\frac{x_1}{x_2}\right)^{5/2}\left[(\alpha^2 + \beta^2) \sin\theta_2 \sinh\phi_2 + \frac{5}{2}\alpha \sin\theta_2 \cosh\phi_2 - \frac{5}{2}\beta \cos\theta_2 \sinh\phi_2\right] + V_2x_1\left(\frac{x_1}{x_2}\right)^{3/2}[-\alpha \sin\theta_2 \cosh\phi_2 + \beta \cos\theta_2 \sinh\phi_2]\right\} \quad (3.11)$$

$$B' = \frac{a}{D_0}\left(\frac{x_1}{a}\right)^2 \frac{1}{\Delta}\left\{M_1\left[\alpha^2 \sin\theta_2 \cos\theta_2 + \alpha\beta \sinh\phi_2 \cosh\phi_2 - \frac{5}{2}\beta \sinh^2\phi_2\right] + V_1x_1[\beta \sinh^2\phi_2] + M_2\left(\frac{x_1}{x_2}\right)^{5/2}\left[-\alpha^2 \sin\theta_2 \cosh\phi_2 - \alpha\beta \cos\theta_2 \sinh\phi_2 - \frac{5}{2}\alpha \sin\theta_2 \sinh\phi_2\right] + V_2x_1\left(\frac{x_1}{x_2}\right)^{3/2}[\alpha \sin\theta_2 \sinh\phi_2]\right\} \quad (3.12)$$

$$C' = \frac{1}{\alpha}\left[-\frac{a^2}{D_0}\left(\frac{x_1}{a}\right)^3\left(V_1 - \frac{5}{2}\frac{M_1}{x_1}\right) - \beta B'\right] \quad (3.13)$$

$$D' = -(a/D_0)(x_1/a)^2 M_1 \quad (3.6')$$

where

$$\Delta \equiv \alpha^2 \sin^2\theta_2 - \beta^2 \sinh^2\phi_2 \quad (3.14)$$

The original constants, A, B, C, and D, in Eq. (2.16) can be written in terms of A', B', C', D' in the following manner:

$$A = (1/b)(4A' + 2\alpha B' - 2\beta C' - \alpha\beta D') \quad (3.15)$$

$$B = (1/b)(2\alpha A' + 4B' - \alpha\beta C' - 2\beta D') \quad (3.16)$$

$$C = (1/b)(2\beta A' + \alpha\beta B' + 4C' + 2\alpha D') \quad (3.17)$$

$$D = (1/b)(\alpha\beta A' + 2\beta B' + 2\alpha C' + 4D') \quad (3.18)$$

where

$$b = 2(\beta^2 + 4)(4 - \alpha^2) \quad (3.19)$$

The 16 influence coefficients for displacements and rotations at x_1 and x_2 are found by substituting the expressions derived for A, B, C, D in Eq. (2.16) and its derivatives evaluated at

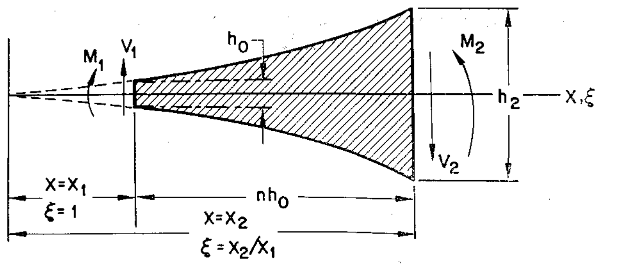

Fig. 2 Longitudinal section through wall with boundary moments and shear forces.

$\xi = 1$ and $\xi = x_2/x_1$. These displacements and rotations can be written as

$$\begin{Bmatrix} w_1 \\ \chi_1 \\ w_2 \\ \chi_2 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix} \begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} \quad (3.20)$$

where

$$\chi_1 = \left.\frac{dw}{dx}\right|_{x_1} \qquad \chi_2 = \left.\frac{dw}{dx}\right|_{x_2} \quad (3.21)$$

It is found that

$$C_{11} = \frac{a^3}{D_0\tau^3}\frac{c^3}{b\Delta}\{-2\alpha^2 \sin^2\theta_2 + 2\beta^2 \sinh^2\phi_2 + \alpha^2\beta \sin\theta_2 \cos\theta_2 - \alpha\beta^2 \sinh\phi_2 \cosh\phi_2\} \quad (3.22)$$

$$C_{12} = -C_{21} = \frac{a^2}{D_0\tau^2}\frac{c^2}{b\Delta}\left\{\alpha^2(1 - \beta^2) \sin^2\theta_2 - \beta^2(1 + \alpha^2) \sinh^2\phi_2 + \frac{5}{2}\alpha\beta^2 \sinh\phi_2 \cosh\phi_2 - \frac{5}{2}\alpha^2\beta \sin\theta_2 \cos\theta_2\right\} \quad (3.23)$$

$$C_{13} = -C_{31} = \frac{a^3}{D_0\tau^3}\left(\frac{x_1}{x_2}\right)^{3/2}\frac{c^3}{b\Delta}\alpha\beta\{\beta \cos\theta_2 \sinh\phi_2 - \alpha \sin\theta_2 \cosh\phi_2\} \quad (3.24)$$

$$C_{14} = C_{41} = \frac{a^2}{D_0\tau^2}\left(\frac{x_1}{x_2}\right)^{5/2}\frac{c^2}{b\Delta}\alpha\beta\left\{(\alpha^2 + \beta^2) \sin\theta_2 \sinh\phi_2 - \frac{5}{2}\beta \cos\theta_2 \sinh\phi_2 + \frac{5}{2}\alpha \sin\theta_2 \cosh\phi_2\right\} \quad (3.25)$$

$$C_{22} = \frac{a}{D_0\tau}\frac{c}{b\Delta}\alpha\beta\left\{-5\alpha\beta(\sin^2\theta_2 + \sinh^2\phi_2) + \beta\left(\alpha^2 + \beta^2 + \frac{25}{4}\right) \sinh\phi_2 \cosh\phi_2 + \alpha\left(\alpha^2 + \beta^2 - \frac{25}{4}\right) \sin\theta_2 \cos\theta_2\right\} \quad (3.26)$$

$$C_{23} = C_{32} = \frac{a^2}{D_0\tau^2}\left(\frac{x_1}{x_2}\right)^{3/2}\frac{c^2}{b\Delta}\alpha\beta\left\{(\alpha^2 + \beta^2) \sin\theta_2 \sinh\phi_2 + \frac{5}{2}\beta \cos\theta_2 \sinh\phi_2 - \frac{5}{2}\alpha \sin\theta_2 \cosh\phi_2\right\} \quad (3.27)$$

$$C_{24} = -C_{42} = \frac{a}{D_0\tau}\left(\frac{x_1}{x_2}\right)^{5/2}\frac{c}{b\Delta}\alpha\beta \times \left\{-\beta\left(\alpha^2 + \beta^2 + \frac{25}{4}\right) \cos\theta_2 \sinh\phi_2 - \alpha\left(\alpha^2 + \beta^2 - \frac{25}{4}\right) \sin\theta_2 \cosh\phi_2\right\} \quad (3.28)$$

Table 1 Influence coefficients for constant-thickness wall

$\tau L/a$	$-C_{11}'^*$	$-C_{12}'^*$	$-C_{13}'^*$	$C_{14}'^*$	$C_{22}'^*$	$-C_{24}'^*$
0.15	6.667	66.668	3.333	66.666	888.946	888.871
0.20	5.000	37.502	2.500	37.499	375.075	374.975
0.25	4.000	24.003	2.000	23.998	192.093	191.968
0.30	3.334	16.671	1.666	16.664	111.223	111.073
0.40	2.501	9.383	1.250	9.370	47.024	46.824
0.50	2.001	6.013	0.999	5.992	24.186	23.936
0.60	1.669	4.186	0.832	4.156	14.112	13.812
0.80	1.255	2.377	0.621	2.324	6.156	5.757
1.00	1.009	1.552	0.493	1.469	3.370	2.873
2.00	0.569	0.567	0.200	0.268	1.076	0.155
4.00	0.500	0.501	-0.002	-0.028	1.002	-0.052
10.00	0.500	0.500	0.000	0.000	1.000	0.000
20.00	0.500	0.500	0.000	0.000	1.000	0.000

$$C_{33} = \frac{a^3}{D_0\tau^3}\left(\frac{x_1}{x_2}\right)^3 \frac{c^3}{b\Delta}\{-2\alpha^2\sin^2\theta_2 + 2\beta^2\sinh^2\phi_2 - \alpha^2\beta\sin\theta_2\cos\theta_2 + \alpha\beta^2\sinh\phi_2\cosh\phi_2\} \quad (3.29)$$

$$C_{34} = -C_{43} = \frac{a^2}{D_0\tau^2}\left(\frac{x_1}{x_2}\right)^4 \frac{c^2}{b\Delta}\left\{\alpha^2(1-\beta^2)\sin^2\theta_2 - \beta^2(1+\alpha^2)\sinh^2\phi_2 - \frac{5}{2}\alpha\beta^2\sinh\phi_2\cosh\phi_2 + \frac{5}{2}\alpha^2\beta\sin\theta_2\cos\theta_2\right\} \quad (3.30)$$

$$C_{44} = \frac{a}{D_0\tau}\left(\frac{x_1}{x_2}\right)^5 \frac{c}{b\Delta}\alpha\beta\left\{-5\alpha\beta(\sin^2\theta_2 + \sinh^2\phi_2) - \beta\left(\alpha^2+\beta^2+\frac{25}{4}\right)\sinh\phi_2\cosh\phi_2 - \alpha\left(\alpha^2+\beta^2-\frac{25}{4}\right)\sin\theta_2\cos\theta_2\right\} \quad (3.31)$$

The quantities c, τ, D_0, α, β, b, Δ, θ, and ϕ are given by Eqs. (2.6, 3.3, 2.12, 2.13, 3.19, 3.14, and 3.10), respectively. The influence coefficients for $c < 1$ are found by replacing β everywhere by $i\beta$. If β is allowed to approach zero, the influence coefficients for the case $c = 1$ result.

4. Influence Coefficients for Constant Wall Thickness

Expressions (3.22–3.31) are complicated indeed, and their derivation is rather tedious. They may be partially checked by comparing them with the influence coefficients for a finite-length cylindrical shell of constant thickness. These quantities can be derived from the solution of Eq. (2.1), where the coefficients D and Eh/a^2 are now constant. The general solution is

$$w = C_1\sin(\tau x/a)\sinh(\tau x/a) + C_2\sin(\tau x/a)\cosh(\tau x/a) + C_3\cos(\tau x/a)\sinh(\tau x/a) + C_4\cos(\tau x/a)\cosh(\tau x/a) \quad (4.1)$$

where τ is given by Eq. (2.6). The four constants of integration are evaluated as before in terms of the edge shear forces

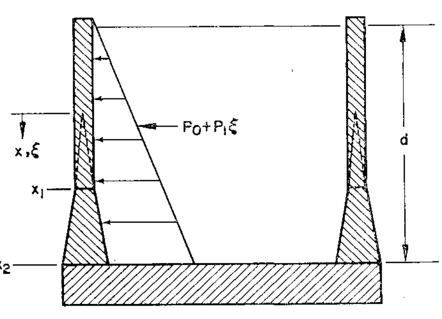

Fig. 3 Longitudinal section through cylindrical fuel tank attached to a rigid bulkhead.

and moments, and the influence coefficients are defined as in Eqs. (3.20). The length of the shell is L. The 16 influence coefficients are

$$C_{11}' = \frac{a^3}{D_0\tau^3}\frac{1}{2\Delta'}\left(\cosh\frac{\tau L}{a}\sinh\frac{\tau L}{a} - \sin\frac{\tau L}{a}\cos\frac{\tau L}{a}\right) = -C_{33}' \quad (4.2)$$

$$C_{12}' = \frac{a^2}{D_0\tau^2}\frac{1}{2\Delta'}\left(\sinh^2\frac{\tau L}{a} + \sin^2\frac{\tau L}{a}\right) = -C_{21}' = C_{34}' = -C_{43}' \quad (4.3)$$

$$C_{13}' = \frac{a^3}{D_0\tau^3}\frac{1}{2\Delta'}\left(\sin\frac{\tau L}{a}\cosh\frac{\tau L}{a} - \cos\frac{\tau L}{a}\sinh\frac{\tau L}{a}\right) = -C_{31}' \quad (4.4)$$

$$C_{14}' = \frac{a^2}{D_0\tau^2}\frac{1}{\Delta'}\left(-\sin\frac{\tau L}{a}\sinh\frac{\tau L}{a}\right) = C_{41}' = C_{23}' = C_{32}' \quad (4.5)$$

$$C_{22}' = \frac{a}{D_0\tau}\frac{1}{\Delta'}\left(-\cos\frac{\tau L}{a}\sin\frac{\tau L}{a} - \cosh\frac{\tau L}{a}\sinh\frac{\tau L}{a}\right) = -C_{44}' \quad (4.6)$$

$$C_{24}' = \frac{a}{D_0\tau}\frac{1}{\Delta'}\left(\sin\frac{\tau L}{a}\cosh\frac{\tau L}{a} + \cos\frac{\tau L}{a}\sinh\frac{\tau L}{a}\right) = -C_{42}' \quad (4.7)$$

where

$$\Delta' \equiv \sin^2\frac{\tau L}{a} - \sinh^2\frac{\tau L}{a} \quad (4.8)$$

5. Influence Coefficients for Slowly Varying Wall Thickness

Equations (3.22–3.31) may be compared with Eqs. (4.2–4.7) for a shell whose wall thickness varies slowly. Let

$$h/h_0 = 1 \quad \text{at} \quad x = x_1 \quad (5.1a)$$

$$h/h_0 = 1 + \epsilon \quad \text{at} \quad x = x_2 \quad (5.1b)$$

$$x_2 - x_1 = L \quad (5.1c)$$

where ϵ is an arbitrarily small number. If

$$h/h_0 = \xi^2 \quad (2.4')$$

then from Eqs. (5.1a–5.1c),

$$x_2/x_1 = 1 + \tfrac{1}{2}\epsilon \quad (5.2)$$

and

$$\frac{x_1}{a} = \frac{2(L/a)}{\epsilon}(1 + O(\epsilon) + \ldots) \quad (5.3)$$

From Eqs. (2.6, 2.12, and 2.13) it is seen that for small ϵ

$$\alpha = \beta = c = \frac{2\tau(L/a)}{\epsilon}(1 + O(\epsilon) + \ldots) \gg 1 \quad (5.4)$$

The quantities (3.10) become

$$\theta_2 = \phi_2 = \frac{2\tau(L/a)}{\epsilon}(1 + O(\epsilon) + \ldots)\log\left(1 + \frac{1}{2}\epsilon + \ldots\right) = \tau L/a(1 + O(\epsilon) + \ldots) \quad (5.5)$$

If Eqs. (5.3–5.5) are substituted into Eqs. (3.22–3.31) and ϵ is allowed to approach zero, the influence coefficients of a shell with parabolic thickness variation approach those for constant thickness [Eqs. (4.2–4.7)].

Dimensionless forms of the influence coefficients for constant thickness are listed in Table 1. They are functions of one parameter $\tau L/a$. Dimensionless forms of the C_{ij} for parabolic thickness variation are listed in Table 2. They are functions of two parameters, h_2/h_0 and c. The quantity h_2/h_0 is the ratio of the thicknesses at x_2 and x_1, respectively, and c is given by (2.6). In both tables, the C_{ij} can be expressed in terms of the dimensionless quantities C_{ij}^* by the following formulas:

$$C_{ij} = C_{ij}^*(a^3/D_0\tau^3) \text{ when } i \text{ and } j \text{ are odd}$$

$$C_{ij} = C_{ij}^*(a^2/D_0\tau^2) \text{ when one subscript is odd and the other is even} \quad (5.6)$$

$$C_{ij} = C_{ij}^*(a/D_0\tau) \text{ when } i \text{ and } j \text{ are even}$$

Table 2 covers all cases of practical interest. If the thickness varies parabolically from h_0 to h_2 within the characteristic length $(ah_0)^{1/2}$, the parameter c satisfies the inequality

$$c \leq \frac{[3(1-\nu^2)]^{1/4}}{(h_2/h_0)^{1/2} - 1} \quad (5.7)$$

For the cases tabulated, the maximum value of c with the constraint (5.7) is $c = 3.14$ (Poisson's ratio $= 0.3$). If c is greater than the right-hand side of Eq. (5.7), the thickness variation is relatively slow, and the influence coefficients approach the appropriate values for a semi-infinite shell of constant thickness. If c is very small and h_2/h_0 is fairly large, the Kirchoff-Love hypothesis (plane sections remain plane) may not be valid, thereby invalidating the governing equation (2.1). In this regard, the influence coefficients listed under $h_2/h_0 = 4$, $c = 0.15$ can be used only for very thin shells. For example, suppose $h_2/h_0 = 4$, $c = 0.15$, and $a/h_0 = 100$. Substitution of these numbers into Eqs. (2.4) and (2.6) with simple algebraic manipulations leads to

$$x_2 - x_1 = 1.16h_0 \quad (5.8)$$

Equation (5.8) states that in a distance approximately equal to the initial thickness of the shell, the wall thickness is quadrupled. Equation (2.1) is obviously not valid for such a rapid variation. On the other hand, if $a/h_0 = 2500$,

$$x_2 - x_1 = 29h_0 \quad (5.9)$$

The thickness quadruples over a distance of 29 times the initial thickness and the Kirchoff-Love hypothesis is, therefore, realistic.

6. Discontinuity Stresses in a Cylindrical Fuel Tank

In a large fuel tank for a missile, discontinuity stresses at a juncture may design the thickness of the wall there. Considerable weight can be saved if the wall is thickened locally in such a way that a minimum amount of material added reduces the discontinuity stresses below some design level. The quantities n and h_2 (see Fig. 2) should be established for a shell of given a/h_0 such that the weight and stress requirements are satisfied. In the following analysis, the discontinuity stresses in a cylindrical fuel tank that is attached to a rigid bulkhead are calculated. Figure 3 shows such a tank filled with liquid to a depth d. The portion of the shell $x < x_1$ is of constant thickness and is assumed to be long compared to the characteristic length $(ah_0)^{1/2}$. It is joined at

Table 2 Influence coefficients for varying thickness

h_2/h_0	c	$-C_{11}^*$	$-C_{12}^*$	$-C_{13}^*$	C_{14}^*	C_{22}^*	C_{23}^*	$-C_{24}^*$	C_{33}^*	$-C_{34}^*$	$-C_{44}^*$
2	0.15	13.229	303.417	5.623	303.417	8773.120	241.674	8773.102	9.393	241.674	8773.110
	0.20	9.920	170.644	4.216	170.643	3700.554	135.919	3700.530	7.044	135.919	3700.540
	0.25	7.935	109.192	3.372	109.192	1894.356	86.972	1894.326	5.634	86.972	1894.338
	0.30	6.613	75.831	2.810	75.830	1096.319	60.399	1096.284	4.695	60.399	1096.298
	0.40	4.959	42.653	2.108	42.652	462.515	33.973	462.468	3.521	33.973	462.486
	0.50	3.967	27.299	1.686	27.297	236.836	21.742	236.777	2.817	21.743	236.800
	0.60	3.306	18.958	1.405	18.956	137.089	15.098	137.018	2.347	15.100	137.046
	0.80	2.480	10.666	1.054	10.662	57.889	8.492	57.795	1.761	8.494	57.832
	1.00	1.984	6.830	0.843	6.823	29.698	5.434	29.580	1.409	5.437	29.627
	2.00	0.995	1.727	0.420	1.699	3.900	1.352	3.664	0.705	1.366	3.757
	4.00	0.520	0.509	0.200	0.398	0.858	0.313	0.393	0.361	0.368	0.575
	6.00	0.404	0.361	0.109	0.133	0.704	0.098	0.047	0.261	0.209	0.297
	10.00	0.406	0.390	0.012	-0.005	0.790	-0.013	-0.033	0.214	0.161	0.230
	20.00	0.451	0.441	-0.000	0.000	0.885	0.001	0.001	0.195	0.142	0.201
	50.00	0.480	0.476	-0.000	0.000	0.952	-0.000	0.000	0.184	0.131	0.186
3	0.15	6.511	82.485	2.547	82.485	1279.206	57.980	1279.182	3.820	57.980	1279.188
	0.20	4.882	46.390	1.910	46.390	539.593	32.608	539.563	2.864	32.608	539.569
	0.25	3.905	29.685	1.527	29.684	276.242	20.865	276.204	2.291	20.866	276.212
	0.30	3.254	20.616	1.273	20.614	159.888	14.490	159.842	1.909	14.490	159.853
	0.40	2.441	11.597	0.955	11.595	67.487	8.150	67.426	1.432	8.151	67.440
	0.50	1.953	7.424	0.764	7.420	34.593	5.216	34.517	1.146	5.217	34.534
	0.60	1.627	5.158	0.636	5.152	20.062	3.621	19.970	0.955	3.623	19.991
	0.80	1.221	2.906	0.477	2.897	8.538	2.036	8.416	0.716	2.039	8.444
	1.00	0.978	1.867	0.381	1.852	4.452	1.301	4.299	0.573	1.306	4.334
	2.00	0.500	0.510	0.187	0.452	0.811	0.315	0.508	0.289	0.334	0.576
	4.00	0.325	0.282	0.070	0.077	0.581	0.046	0.015	0.159	0.111	0.135
	6.00	0.349	0.329	0.014	0.002	0.682	-0.008	-0.020	0.131	0.084	0.100
	10.00	0.405	0.389	-0.002	-0.002	0.787	-0.002	0.000	0.117	0.071	0.083
	20.00	0.451	0.441	0.000	0.000	0.885	-0.000	0.000	0.106	0.063	0.073
	50.00	0.480	0.476	0.000	0.000	0.952	0.000	0.000	0.100	0.058	0.067
4	0.15	4.263	38.961	1.581	38.961	427.827	25.210	427.801	2.200	25.210	427.805
	0.20	3.197	21.912	1.186	21.912	180.480	14.178	180.447	1.650	14.178	180.452
	0.25	2.557	14.022	0.948	14.021	92.412	9.072	92.370	1.320	9.072	92.376
	0.30	2.131	9.739	0.790	9.737	53.505	6.300	53.454	1.100	6.301	53.462
	0.40	1.598	5.479	0.593	5.476	22.614	3.543	22.546	0.825	3.544	22.557
	0.50	1.279	3.509	0.474	3.504	11.624	2.267	11.540	0.660	2.269	11.553
	0.60	1.066	2.440	0.395	2.433	6.775	1.574	6.674	0.550	1.576	6.689
	0.80	0.800	1.380	0.296	1.367	2.943	0.884	2.808	0.413	0.887	2.829
	1.00	0.642	0.893	0.236	0.873	1.598	0.564	1.430	0.330	0.569	1.455
	2.00	0.341	0.284	0.113	0.206	0.487	0.130	0.154	0.168	0.149	0.204
	4.00	0.292	0.266	0.029	0.020	0.573	0.004	-0.011	0.099	0.059	0.063
	6.00	0.348	0.330	-0.001	-0.004	0.680	-0.008	-0.007	0.085	0.047	0.049
	10.00	0.405	0.389	0.000	0.000	0.787	0.000	0.000	0.076	0.040	0.041
	20.00	0.451	0.441	0.000	0.000	0.885	0.000	0.000	0.069	0.035	0.036
	50.00	0.480	0.476	0.000	0.000	0.952	0.000	0.000	0.060	0.030	0.031

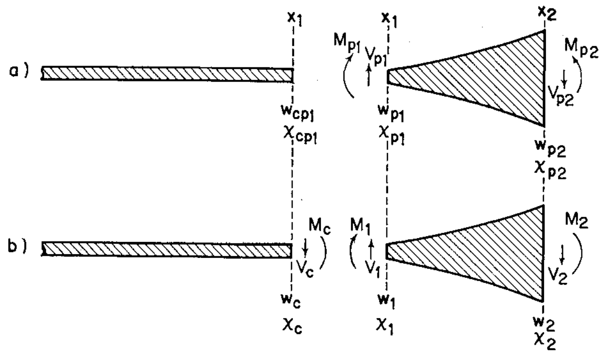

Fig. 4 a) Exploded longitudinal section through cylindrical tank showing moments and shear forces corresponding to particular solutions and b) additional moments and shear forces needed for compatibility at x_1 and x_2.

$x = x_1$ to a short shell of parabolic thickness variation. The governing equation in the region $x_1 < x \leq x_2$ is

$$(\xi^6\delta'')'' + 4c^4\xi^2\delta = 4c^4(a/h_0)(1/E)(P_0 + P_1\xi) \quad (6.1)$$

and in the region $x < x_1$ is

$$\delta_c'''' + 4c^4\delta_c = 4c^4(a/h_0)(1/E)(P_0 + P_1\xi) \quad (6.2)$$

where

$$P_0 \equiv \gamma(x_2 - d) \qquad P_1 \equiv -\gamma x_1 \quad (6.3)$$

The constant γ is the specific gravity of the liquid. The subscript c denotes constant thickness. The pressure is positive inward. The particular solutions of Eq. (6.1) and (6.2) are easily found to be, respectively,

$$\delta_p = \frac{c^4}{3 + c^4}\frac{a}{h_0}\frac{1}{E}(P_0\xi^{-2} + P_1\xi^{-1}) \quad (6.4)$$

$$\delta_{cp} = \frac{a}{h_0}\frac{1}{E}(P_0 + P_1\xi) \quad (6.5)$$

The moment and shear distributions corresponding to the particular solutions of Eq. (6.1) and (6.2) are

$$M_p = -\frac{D_0 a}{x_1^2}\frac{c^4}{3 + c^4}\frac{a}{h_0}\frac{1}{E}(6P_0\xi^2 + 2P_1\xi^3) \quad (6.6)$$

$$V_p = \frac{-D_0 a}{x_1^3}\frac{c^4}{3 + c^4}\frac{a}{h_0}\frac{1}{E}(12P_0\xi + 6P_1\xi^2) \quad (6.7)$$

$$M_{cp} = 0 \quad (6.8)$$

$$V_{cp} = 0 \quad (6.9)$$

The particular solutions (6.4) and (6.5) and the corresponding moment and shear force (6.6) and (6.7) are shown in Fig. 4a. The displacement w is defined by Eq. (2.3). It is evident that the displacements, rotations, moments, and shears (6.4–6.9) are not compatible at x_1, and that the boundary conditions $w = \chi = 0$ at x_2 are not satisfied. Figure 4b shows the moments and shears that must be introduced at x_1 and x_2 in order to satisfy all conditions. Geometric compatibility at x_1 demands that

$$c_{cp1} + w_c = w_{p1} + w_1 \quad (6.10)$$

$$\chi_{cp1} + \chi_c = \chi_{p1} + \chi_1 \quad (6.11)$$

Physical compatibility at x_1 demands that

$$M_c = M_{p1} + M_1 \quad (6.12)$$

$$V_c = V_{p1} + V_1 \quad (6.13)$$

The clamped edge condition at x_2 requires that

$$w_{p2} + w_2 = 0 \quad (6.14)$$

$$\chi_{p2} + \chi_2 = 0 \quad (6.15)$$

The deformation quantities w_1, χ_1, w_2, and χ_2 are expressed in terms of M_1, V_1, M_2, and V_2 by Eqs. (3.20), and w_c and χ_c are expressed in terms of M_c and V_c through the equations

$$\begin{Bmatrix} w_c \\ \chi_c \end{Bmatrix} = [\kappa_{ij}]\begin{Bmatrix} V_c \\ M_c \end{Bmatrix} \quad (6.16)$$

The κ_{ij} are the influence coefficients for a semi-infinite cylindrical shell with constant wall thickness:

$$\kappa_{11} = \frac{1}{2}\frac{a^3}{D_0\tau^3} \qquad \kappa_{12} = -\kappa_{21} = -\frac{1}{2}\frac{a^2}{D_0\tau^2}$$
$$\kappa_{22} = -\frac{a}{D_0\tau} \quad (6.17)$$

Now M_c, V_c, M_1, V_1, M_2, and V_2 can be determined from the relations (6.10) to (6.15). The moment distribution in the region $x_1 < x < x_2$ due to M_1, V_1, M_2, and V_2 is found with the help of Eqs. (3.6–3.9) and Eq. (3.4). The hoop stress distribution is calculated from

$$N = -Eh\delta = -Eh_0\xi^2\delta \quad (6.18)$$

where δ is given by Eq. (2.16). The resultant moment and hoop stress distributions in $x_1 < x < x_2$ are the sums of the distributions due to M_1, V_1, M_2, and V_2 and those due to δ_p.

The moment and hoop stress distributions in the constant thickness region ($x < x_1$) are obtained from formulas in Timoshenko[8] once M_c and V_c are known. The stresses at the outer fiber of the shell are computed from the moment and hoop resultants by means of

$$\sigma_x|_{-h/2} = 6M/h^2 \quad (6.19)$$

$$\sigma_\theta|_{-h/2} = N/h - \nu 6M/h^2 \quad (6.20)$$

Figures 5 and 6 demonstrate the effect of edge thickening on the discontinuity stresses (6.19) and (6.20). For various degrees of edge thickening, the axial and circumferential stress parameters $\sigma_x/a\gamma$ and $\sigma_\theta/a\gamma$ are plotted vs the nondimensional distance from the edge y/a. These stresses are compared with the stresses generated in a similar shell

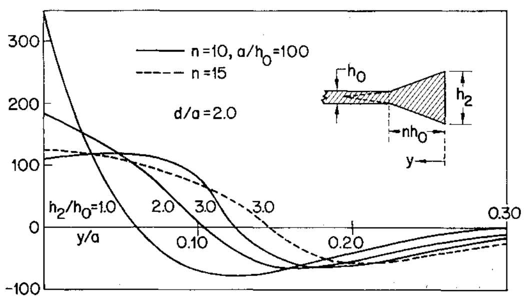

Fig. 5 Axial stress reduction due to thickening of the wall.

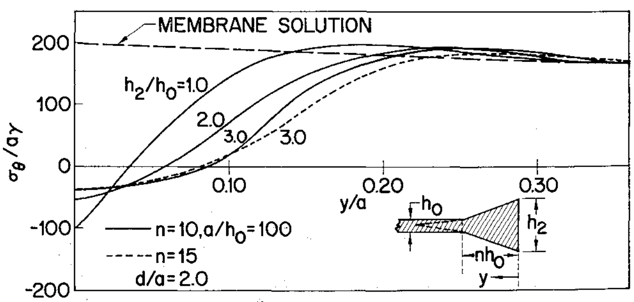

Fig. 6 Circumferential stress reduction due to thickening of the wall.

of constant thickness (lines labeled $h_2/h_0 = 1.0$). The depth of the liquid is equal to the diameter of the shell. For the case $a/h_0 = 100$, $n = 10$, and $h_2/h_0 = 3$, the maximum $\sigma_x/a\gamma$ is reduced from 350 to 120 (Fig. 5). This is indeed a significant reduction and demonstrates clearly that the influence coefficients given by Eqs. (3.22–3.31) must be used to obtain an accurate estimate of the maximum stress. The maximum $\sigma_\theta/a\gamma$ is reduced from 197 to 188 (Fig. 6). The effect of the thickening on $\sigma_{\theta\ \max}$ is small because the hoop stress builds up to its maximum away from edge where the shell thickness is constant.

In all the cases plotted, the maximum stress for the shells with the thickened edges is slightly higher than the membrane hoop stress due to the hydrostatic pressure.

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